SOME CHARACTERIZATIONS FOR THE DIMENSION OF ORDERED MULTISETS

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Abstract
The paper presents a study of dimension as an important combinatorial parameter of ordered multisets defined over partially ordered base sets. The relationship between the dimension of a partially ordered multiset and that of the underlying generic set is investigated and some results are presented.

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Introduction
The notion of dimension is recognized as the most widely studied combinatorial parameter in the theory of ordered sets (Felsner et al., 2015; Joret et al., 2016; Streib and Trotter, 2014; Trotter, 2017). The dimension of a partially ordered set (poset) is the minimum number of linear extensions needed to characterize it. Duchin and Miller (1941) defined the dimension of a poset \( P \), denoted \( \dim(P) \), as the least positive integer \( d \) for which \( P \) has a realizer of size \( d \). An extension of this concept naturally arises if the linear orders which form a realizer of \( P \) are required to have certain additional properties (see Kelly (1981), Trotter (1992) and Trotter (1995) for extensive background information on the theory of posets and their dimension). Several characterizations have been established for the dimension of a poset. It can be deduced from Dilworth’s decomposition theorem (Dilworth, 1950) that for any poset \( P \), \( \dim(P) \leq w \), where \( w \) is the width of \( P \). Hiraguchi (1951) proved that \( \dim(P) \leq [N/2] \), where \([N]\) is the cardinality of the ground set. In Kierstead and Trotter (1985), several inequalities bounding the greedy dimension of a poset \( P \) as a function of other parameters of \( P \) were developed. The notion of super greedy dimension was studied in Kierstead et al., (1987). There has been renewed interest in studying combinatorial properties of a poset determined by geometric properties of its order diagram and topological properties of its cover graph. An early result that has motivated research in this direction is the work of Trotter and Moore (1977). Recent results characterizing the dimension of a poset using these properties include, Felsner et al., (2015), Trotter and Wang (2015), Joret et al., (2016) and Trotter (2017). In Joret et al., (2016), it was shown that the dimension of a finite poset is bounded in terms of its height and the tree-width of its cover graph. Felsner et al., (2015) proved that the dimension of a poset is bounded if its cover graph is outerplanar. In Trotter and Wang (2015), it was proved that the dimension of a planar poset with \( \tau \) minimal element is at most \( 2\tau + 1 \) and that this bound is tight for \( \tau = 1 \) and \( \tau = 2 \). For \( \tau \geq 3 \), there exist planar ordered sets with \( \tau \) minimal elements having dimension sets having come to an age.

In view of numerous applications of multisets (msets) found in both hard and soft sciences (Blizard, 1989), Singh and Isah, (2016), Singh et al., (2007) and Wildberger (2003), are excellent expositions on mssets), researchers have shown interest in extending concepts and results on ordered sets to mssets (Kilbarada and Jovovi, 2004; Conder et al., 2007; Girish and Sunil 2009). This paper focuses on explicating the concept of dimension of a partially ordered multiset (pomset). Results on the dimension of a poset are generalized using an ordering that introduces hierarchies between the points of a finite msset.

Multisets
When repetition of elements of a set is significant, the mathematical structure obtained is called an msset. If \( S = \{x_1, x_2, \ldots, x_r\} \) is a set, an mset \( M \) over \( S \) is a cardinal-valued function i.e., \( M : S \rightarrow N \) such that \( x \in \text{Dom}(M) \) implies \( M(x) \) is a cardinal number and \( M(x) = m(x) > 0 \), where \( m(x) \) denotes the frequency (usually called multiplicity) of the object \( x \) in \( M \). The set \( S \) represents the ground set of the mset \( M \). In this work, for an element \( x \in M \) and its multiplicity \( m, mx \) will represent a point in \( M \). So that \( M = \{m_1, m_2, m_3, \ldots, m_n\} \), where \( M(x) = m_x \) with \( x \in [1, n] \). The set \( \{x \in S | M(x) > 0\} \) is called the root set of \( M \), and it is denoted \( M^* \). Elements of the root set represent objects in an mset, and each individual occurrence of an object is called an element of the mset. The cardinality of an mset is the sum of the multiplicities of all its distinct elements.

The class of all finite mssets containing objects from a set \( S \) will be denoted by \( M(S) \). For two mssets \( M, N \) in \( M(S) \), \( M \) is a subset of \( N \), denoted \( M \subseteq N \), if \( M(x) \leq N(x) \) for all \( x \in S \) and \( M \) is strictly contained in \( N \) if and only if \( M(x) < N(x) \) for at least one \( x \). The null mset, denoted \( \emptyset \), which is contained in every msset is given by, \( \emptyset(x) = 0, \forall x \in S \). A submset is called whole if it contains all multiplicities of common elements from the parent set. A full submset contains all objects of the parent mset. Clearly, every msset contains a unique full submset, its root set. For any two mssets \( M \) and \( N \), if \( M \subseteq N \) and \( \text{Dom}(M) = \text{Dom}(N) \), then \( M \) is a full submset of \( N \).

The union of two mssets \( M \) and \( N \) is the mset given by, \( (M \cup N)(x) = \max\{m,n\} \) such that \( mx \in M \) and \( nx \in N \), for all \( x \in S \). Their intersection is the mset given by \( (M \cap N)(x) = \min\{m,n\} \), for all \( x \in S \). The union \( M \cup N \) is defined as the mset containing \( m + n \) occurrences of any object occurring \( m \) times in \( M \) and \( n \) times in \( N \). An mset \( M \) is finite if both the number of objects in \( M \) and their multiplicities are finite. The mssets dealt with in this work are finite mssets with nonnegative integral multiplicities.
Partially Ordered Multisets (Pomsets)

We present some basic definitions and notations on the ordered mset structure in order to make the article self-contained (see Balogun and Tella (2017) for details). A point in an mset will be denoted by \( m_{x_i} \) where, \( m_i \) is the multiplicity of \( x_i \) in \( M \). Let \( M \) be an mset defined over a partially ordered base set \( P = (S, \leq) \), with points \( m_{x_1}, m_{x_2}, \ldots, m_{x_n} \). For any pair \( m_i, m_j \) and \( m_{x_i}, m_{x_j} \) in \( P \), with \( i, j \in \{1, 2, \ldots, n\} \), \( m_{x_i} \leq m_{x_j} \) if and only if \( x_i \leq x_j \) in \( P \). The points \( m_{x_i} \) and \( m_{x_j} \) coincide if and only if \( x_i = x_j \) (principle of uniqueness of multiplicity of an object).

The points \( m_{x_i} \) and \( m_{x_j} \) are comparable if and only if, \( m_{x_i} \leq m_{x_j} \lor m_{x_j} \leq m_{x_i} \) (denoted \( m_{x_i} \equiv m_{x_j} \)). They are incomparable (denoted \( \overline{m_{x_i} \equiv m_{x_j}} \)).

The ordering \( \leq \) is called a partial mset order or simply an mset order if it is reflexive, antisymmetric, and transitive, and a linear mset order (or a total mset order) if it is a partial mset order, and all points \( m_{x_i}, m_{x_j} \) in \( M \) are comparable under \( \leq \).

A pomset is a pair \( M = (M, \leq) \), where \( \leq \) is a partial order on the mset \( M \). The strict order associated with \( \leq \) will be denoted by \( < \). For any \( m_{x_i} < m_{x_j} \) implies \( m_{x_i} \neq m_{x_j} \).

The dual of the pomset \( M \) is the pomset, denoted \( M^d \), with \( m_{x_i} \leq m_{x_j} \iff m_{x_j} \leq m_{x_i} \) for all \( x_i, x_j \) in \( M \).

Given any mset \( M \in M(S) \) defined over a partially ordered base set \( P = (S, \leq) \), it can be verified that \( M = (M, \leq) \) is a pomset. For a pomset \( M \), a point \( m_{x_i} \) in \( M \) is maximal, respectively minimal, when there is no point \( m_{x_j} \) with \( m_{x_i} < m_{x_j} \), respectively \( m_{x_i} > m_{x_j} \). If \( M \) has a unique maximal, respectively minimal, point, then it is called the maximum, respectively minimum, point of \( M \).

A suborder \( \leq_X \) is the restriction of \( \leq \) to pairs of points in \( N \subseteq M \) such that, \( x_i, x_j \leq \leq_X x_i \iff m_{x_i} \leq_X m_{x_j} \), where \( m_{x_i}, m_{x_j} \in N \) and \( m_{x_i}, m_{x_j} \in M \). The pair \( (N, \leq_X) \) which we simply denote by \( \mathcal{N} \), is called a subpomset of \( \mathcal{M} \). A subpomset \( \mathcal{C} = (N, \leq_X) \) of a pomset \( \mathcal{M} \) is an mset chain if \( \mathcal{C} \) is linearly (or totally) ordered, i.e., \( m_{x_i} < m_{x_j} \) for all pairs of points \( x_i, x_j \in N \). Also, a subpomset \( \mathcal{A} = (L, \leq_A) \) of \( \mathcal{M} \) is called an mset antichain if, \( l_i < l_j \) for all pairs \( l_i, l_j \in L \). An mset chain \( \mathcal{C} \) in a pomset \( \mathcal{M} \) is maximal if it is not strictly contained in any other mset chain of \( \mathcal{M} \), and maximum, if it contains maximum number of points. Maximal and maximum mset antichains are defined analogously.

The height of \( \mathcal{M} \) is the number of points in a maximum mset chain and the width of \( \mathcal{M} \) is the number of points in a maximum mset antichain.

**Dimension of a Pomset**

**Definition 1**

For an mset \( M \in M(S) \), let \( \mathcal{M} = (M, \leq_M) \) and \( \mathcal{N} = (M, \leq_M) \) be two pomsets defined over partially ordered base sets \( P \) and \( Q \), respectively, where \( P \) and \( Q \) have the same ground set. Then \( \mathcal{N} \) is an mset extension of \( \mathcal{M} \) if \( m_{x_i} \leq_M m_{x_j} \) implies that \( x_i \leq_M x_j \). The relation \( \leq_M \) is contained in \( \leq_M \) and hence the mset extension of \( \mathcal{M} \) that is a linear mset order is called an mset linear extension of \( \mathcal{M} \).

**Example 1**

Let \( \mathcal{M} = (M, \leq_M) \) be a pomset where, \( M = \{ x_1, x_2, x_3, x_4, x_5, x_6 \} \). Suppose that the mset order on \( M \) is defined as follows:

\[
\begin{align*}
3x_1 & \leq_M 3x_2 \leq_M 3x_3 \\
3x_1 & \leq_M 2x_3 \leq_M 3x_4 \\
3x_1 & \leq_M 2x_3 \leq_M 3x_5 \leq_M 3x_7 \\
3x_1 & \leq_M 2x_3 \leq_M 3x_5 \leq_M 3x_6 \\
3x_1 & \leq_M 2x_3 \leq_M 3x_5 \leq_M 3x_6 \leq_M 3x_7 \\
\end{align*}
\]

Then, the sets \( \ell_1 = \{ x_3, x_5, x_2, x_3, x_5, x_4, x_3, x_7 \} \) and \( \ell_2 = \{ x_3, x_2, x_4, x_5, x_3, x_7 \} \) with, \( 3x_1 \leq_\ell \leq_M 3x_2 \leq_\ell 2x_3 \leq_\ell 4x_6 \leq_\ell 3x_7 \leq_\ell 5x_2 \leq_\ell 3x_4 \leq_\ell 4x_6 \leq_\ell 3x_7 \leq_\ell 5x_2 \leq_\ell 3x_4 \leq_\ell 4x_6 \leq_\ell 3x_7 \).

Mset Realizers

Let \( \mathcal{J} = \{ m_{x_i}, m_{x_j} \in M : m_{x_i} \leq_M m_{x_j} \} \). For any pair \( m_{x_i}, m_{x_j} \) in \( \mathcal{J} \), there is a corresponding pair \( x_i \leq x_j \) in \( P \) (by the definition of \( \leq_M \)). By Szpilrajn’s extension theorem (Szpilrajn, 1930) for ordered sets, there exist two linear extensions \( \ell_1, \ell_2 \) with \( 3x_1 \leq_\ell 3x_2 \leq_\ell 3x_3 \leq_\ell 3x_4 \leq_\ell 3x_5 \leq_\ell 3x_6 \leq_\ell 3x_7 \).

**Theorem 1**

Let \( \mathcal{M} \) be a pomset defined over a poset \( P \). If \( R = \{ \ell_1, \ldots, \ell_r \} \) and \( U = \{ l_1, \ldots, l_s \} \) are realizers for \( \mathcal{M} \) and \( P \), respectively, then \( \mathcal{M} \) has points \( s \leq r \).

**Proof**

Given a pomset \( \mathcal{M} = (M, \leq_M) \) where \( M = \{ m_{x_i}, m_{x_j}, \ldots, m_{x_n} \} \). Let \( \mathcal{J} = \{ m_{x_i}, m_{x_j} \in M : m_{x_i} \leq_M m_{x_j} \} \). For any pair \( m_{x_i}, m_{x_j} \) in \( \mathcal{J} \), there exist linear extensions \( \ell_i, \ell_j \) with \( m_{x_i} \leq_\ell m_{x_j} \). This implies \( \mathcal{M} \) is a linear mset order. Now, for any pair \( m_{x_i}, m_{x_j} \) in \( \mathcal{J} \) such that \( m_{x_i} \leq_M m_{x_j} \), there is a corresponding pair \( x_i \leq x_j \) in \( (S, \leq) \). Similarly, \( s \leq 2m \) for \( m \) incomparable in \( P \). Let \( U = \{ l_i : x_i \in \mathcal{U} \} \). If each pair in \( U \), \( x_i \leq x_j \) in \( (S, \leq) \). Then, \( s = m \). Since each mset linear order \( \ell_i \) is induced by the linear order \( l_i \), this implies \( \ell = \ell \).

**SOME CHARACTERIZATIONS,…**
The concept of dimension was defined on a partially ordered multiset structure. The relationship between the dimension of a pomset $M$ and that of its underlying generic set was investigated and presented via some results. An mset is an extended notion of a set, hence, studying the concept of dimension on an ordered mset structure leads to generalizations. The problem of extending the concepts of greedy and super greedy dimensions studied in Kierstead and Trotter (1985), and Kierstead et al., (1987) via the ordered mset structure used in this work seems promising.

References


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